18.152 PROBLEM SET 3

due April 1st 9:30 am

You can collaborate with other students when working on problems. However, you should write the solutions using your own words and thought.

Problem 1. *Suppose that a smooth function* $u : \mathbb{R}^3 \to \mathbb{R}$ *satisfies* $\Delta u \geq 6$. *Show that given* $R > 0$ *the following holds.*

$$u(\vec{0}) \leq -\frac{3}{5}R^2 + \frac{3}{4\pi R^3}\int_{B_R(\vec{0})} u(\vec{y})d\vec{y}.$$

Hint: $\Delta|\vec{x}|^2 = 6$.

Problem 2. *Let* $\Omega = B_1(0) \subset \mathbb{R}^2$. *Suppose that a smooth function* $u : \overline{\Omega} \to \mathbb{R}$ *satisfies*

$$\int_\Omega 2\left|\nabla u \cdot \tfrac{(1,1)}{\sqrt{2}}\right|^2 + \left|\nabla u \cdot \tfrac{(1,-1)}{\sqrt{2}}\right|^2 d\vec{x} \leq \int_\Omega 2\left|\nabla v \cdot \tfrac{(1,1)}{\sqrt{2}}\right|^2 + \left|\nabla v \cdot \tfrac{(1,-1)}{\sqrt{2}}\right|^2 d\vec{x},$$

for any smooth function $v : \overline{\Omega} \to \mathbb{R}$ *such that* $u = v$ *holds on* $\partial\Omega$. *Show that* $0 = 3u_{11} + 2u_{12} + 3u_{22}$ *holds in* $\overline{\Omega}$.

Hint: Consider $\hat{u}(x_1, x_2) = u\Big(\frac{x_1+x_2}{\sqrt{2}}, \frac{x_1-x_2}{\sqrt{2}}\Big)$.

Problem 3. *Let a smooth function* $u : \mathbb{R}^2 \setminus B_1(0) \to \mathbb{R}$ *be harmonic. Find all solutions* $u(r\cos\theta, r\sin\theta)$ *satisfying*

$$u(\cos\theta, \sin\theta) = \cos(2\theta), \qquad \lim_{r\to+\infty} \tfrac{1}{r}u(r\cos\theta, r\sin\theta) = 0.$$

Problem 4. *Show that the Green function to a smooth bounded domain* Ω *is symmetric. Namely,*

$$G(x, y) = G(y, x).$$

Hint: Given $x \neq y$, choose $\epsilon < |x - y|$. Next, define $u(z) = G(x, z)$ and $v(z) = G(y, z)$, and apply the Green's identity over the domain $\Omega \setminus (B_\epsilon(x) \cup B_\epsilon(y))$. By passing $\epsilon \to 0$, show $u(y) = v(x)$.